A Complete Classification of 2×2 Linear Iterative Systems

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Abstract

The study of 2×2 linear iterative systems is incorporated in many books on ordinary differential equations. As in the case of linear systems of differential equations, the classification of the equilibrium solution $(0,0)$ leads to an analysis of the eigenvalues and eigenvectors of the system matrix. However the authors do not know of any textbook that investigates the phase portraits for the many borderline cases in the trace-determinant Plane. The purpose of this paper is to fill in these details. In addition, a recent software developed by Hubert Hohn of Massachusetts College of Art for the purpose of this investigation is used for pictorial illustrations of these portraits.

1 Systems of Iterative Equations

For linear systems of differential equations $\frac{d\vec{Y}}{dt} = A\vec{Y}(t)$, or $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, we seek solutions of the form $\vec{Y}(t) = e^{\lambda t}\vec{v}$, where λ is an eigenvalue for the system matrix A and $\vec{v}$ is any corresponding eigenvector. One motivation for seeking such a solution is the fact that a natural guess for a second-order homogeneous linear differential equation $ay'' + by' + cy = 0$ is an exponential function of the form $e^{\lambda t}$ where λ is a solution to the auxiliary equation $ar^2 + br + c = 0$.
In the discrete case, the general solution of a second-order homogeneous iterative equation $ay_{n+2} + by_{n+1} + cy_n = 0$, where y_n is a sequence, the general solution takes one of the following 3 forms:

1. $y_n = c_1{\lambda_1}^n + c_2{\lambda_2}^n$, where λ_1 and λ_2 are two distinct real solutions of the auxiliary equation $ar^2 + br + c = 0$.

2. $y_n = c_1{\lambda_1}^n + c_2 n{\lambda_1}^n$, where λ_1 is a double root for the auxiliary equation.
3. $y_n = r^n\left(c_1\cos(n\theta) + c_2\sin(n\theta)\right)$, where $\lambda = re^{i\theta}$ is a complex root for the auxiliary equation.

For the linear homogeneous iterative system

$$\begin{cases} x_{n+1} = ax_n + by_n \\ y_{n+1} = cx_n + dy_n \end{cases}$$

or in vector form

$$\vec{Y}_{n+1} = A\vec{Y}_n,$$

where $\vec{Y_n} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(0,0)$ is clearly an equilibrium solution. Using the information from the second-order iterative equations, one seeks other solutions of the form $\vec{Y_n} = \lambda^n\vec{v}$, where λ is an eigenvalue for matrix A and $\vec{v}$ is a corresponding eigenvector. In the case where matrix A has two distinct eigenvalues (real or complex), the general solution $\vec{Y_n}$ takes the form: $\vec{Y_n} = c_1\lambda_1^n\vec{v_1} + c_2\lambda_2^n\vec{v_2}$, where $\vec{v_1}, \vec{v_2}$ are corresponding eigenvectors for λ_1 and λ_2 respectively (see Farlow, Hall, McDill, and West [2]). For the complex eigenvalues, the corresponding eigenvectors have complex entries and the qualitative behavior of the solution is better understood by noticing that the real and imaginary parts of any one solution are themselves solutions to the system.

Below is an example of a completely decoupled linear iterative system that can be solved with or without any reference to the eigenvalues of the corresponding system matrix.

Example 1.1 *Consider the system*

$$\begin{cases} x_{n+1} = 1.05x_n \\ y_{n+1} = 1.1y_n \end{cases}$$

Its solution is clearly

$$\begin{cases} x_n = x_0(1.05)^n \\ y_n = y_0(1.1)^n. \end{cases}$$

Using vector notation, the system takes the form

$$\vec{Y}_{n+1} = \begin{pmatrix} 1.05 & 0 \\ 0 & 1.1 \end{pmatrix}\vec{Y}_n.$$

Its eigenvalues are 1.05 *and* 1.1*, and two corresponding eigenvectors are respectively* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. *Thus the general solution takes the form:*

$$\vec{Y_n} = c_1(1.05)^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2(1.1)^n\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or equivalently,

$$\begin{cases} x_n = c_1(1.05)^n \\ y_n = c_2(1.1)^n. \end{cases}$$

Thus as n tends to ∞, both x_n and y_n approach infinity. The origin $(0,0)$ in this case is called a ***source*** *(See Figure 1). However we note in this case that the y-components are increasing faster than the x-components; consequently, if we were to "connect" the points that form the iteration (a helpful device that is not actually part of the trajectory or orbit in the discrete case), the resulting polygonal curve tends to become "parallel" to the y-axis, i.e. the eigenline corresponding to the larger eigenvalue, just as for systems of differential equations.*

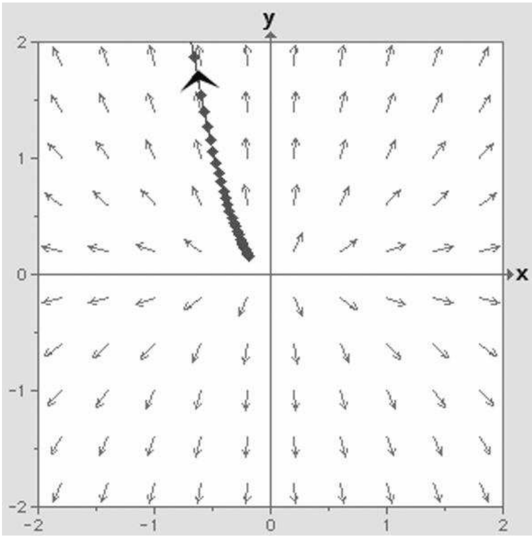

Figure 1: A Source

In this example as well as in all subsequent examples, a software recently developed by Hu Hohn for the purpose of this investigation is being used. Hohn is the director of Computer Arts at Massachusetts College of Art, Boston, MA, and has been a software designer for many of the CD's that appear in contemporary math books. Particularly, Hohn worked closely with author Jean-Marie McDill on a companion CD for the book Interactive Differential Equation (IDE), co-authored by John Cantwell, Steven Strogatz, and Beverly West ([5]). IDE is a collection of 92 labs built each to offer a complete understanding of a particular concept in ordinary differential equations. One tool in particular is concerned with the linear classification of the equilibrium solution $(0,0)$ for 2×2 linear systems of differential equations. The usefulness of this tool led the authors to ask Hohn to develop a similar one for 2×2 linear iterative systems. Figure 2 below is a snapshot of the classification tool for the continuous case of example 1, as it will appear in IDE.

The classification tool developed by Hohn for the discrete case follows the same guidelines as in the continuous case. The window consists of a system matrix in which the user can input the entries of the system matrix, the eigenvalues and eigenlines of the system matrix, the trace and determinant of the matrix and the trace-determinant plane (whose importance will become clear later), and the phase plane with or without the vector field. The latter object adds a visual dimension to the solution behavior as time progresses allowing a classification of the origin (e.g. stable vs. unstable). In addition, the list of iterated values appears on the right hand side of the tool (see figure 3).

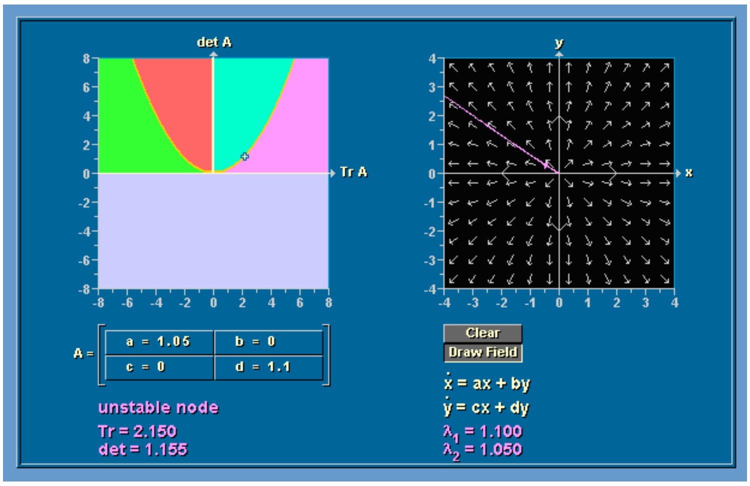

Figure 2: Classification Tool in IDE

Below are 3 other examples in the discrete case with a detailed view of the newly developed (yet non commercialized) tool in example 1.2.

Example 1.2 *Consider the system*

$$\begin{cases} x_{n+1} = 1.05 y_n \\ y_{n+1} = -1.1 x_n \end{cases}$$

or

$$\vec{Y}_{n+1} = \begin{pmatrix} 0 & 1.05 \\ -1.1 & 0 \end{pmatrix} \vec{Y}_n.$$

The eigenvalues of this system are the solutions to the equation $\lambda^2 + 1.155 = 0$*. The solutions are the pure imaginary numbers* $\lambda = \pm i\sqrt{1.155} \approx \pm 1.075i$*, and* $\vec{v_1} = \begin{pmatrix} 1 \\ i\frac{1.075}{1.05} \end{pmatrix}$*, and* $\vec{v_2} = \begin{pmatrix} 1 \\ -i\frac{1.075}{1.05} \end{pmatrix}$ *are two linearly independent eigenvectors. One solution is therefore:*

$$\lambda_1^n \vec{v_1} = (1.075i)^n \begin{pmatrix} 1 \\ i\frac{1.075}{1.05} \end{pmatrix}.$$

But $(1.075i)^n = (1.075)^n \left[\cos\left(n\frac{\pi}{2}\right) + i\sin\left(n\frac{\pi}{2}\right)\right]$*. Separating the real and imaginary parts, we obtain:*

$$(1.075)^n \begin{pmatrix} \cos\left(n\frac{\pi}{2}\right) \\ -\frac{1.075}{1.05}\sin\left(n\frac{\pi}{2}\right) \end{pmatrix} + i(1.075)^n \begin{pmatrix} \sin\left(n\frac{\pi}{2}\right) \\ \frac{1.075}{1.05}\cos\left(n\frac{\pi}{2}\right) \end{pmatrix}.$$

Since the real and imaginary parts are also linearly independent solutions to this system, the general solution takes the form:

$$\vec{Y_n} = c_1(1.075)^n \begin{pmatrix} \cos\left(n\frac{\pi}{2}\right) \\ -\frac{1.075}{1.05}\sin\left(n\frac{\pi}{2}\right) \end{pmatrix} + c_2(1.075)^n \begin{pmatrix} \sin\left(n\frac{\pi}{2}\right) \\ \frac{1.075}{1.05}\cos\left(n\frac{\pi}{2}\right) \end{pmatrix}.$$

The sine and cosine terms in the solutions produce a spiralling effect on the solution. Due to the fact that the magnitude of λ *is bigger than one, the solution will spiral outwards, away from the origin. The origin in this case* ($||\lambda|| > 1$) *is called a* ***spiral source****. (See Figure 3).*

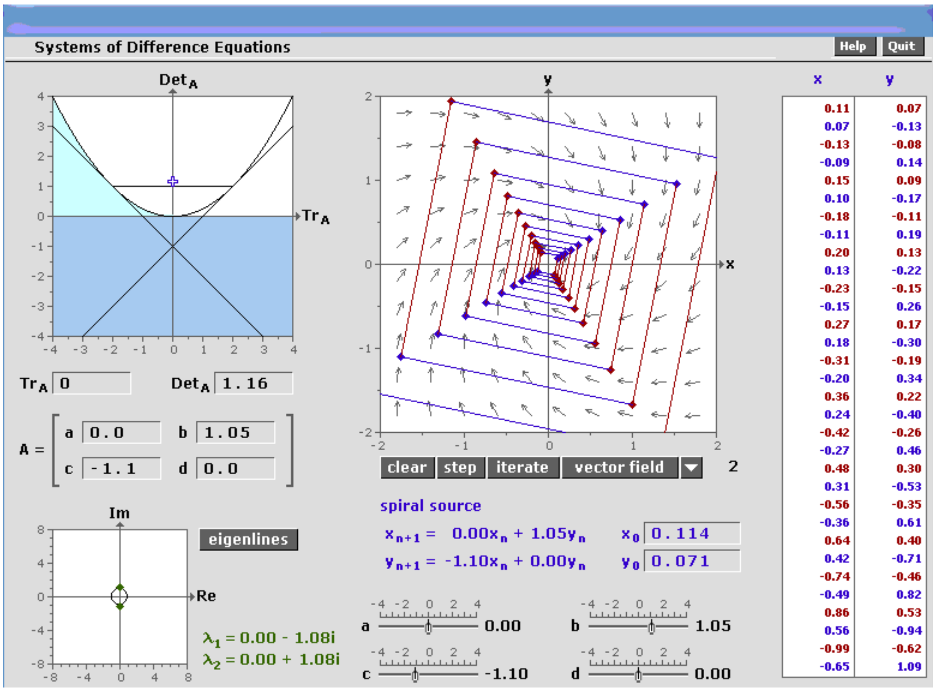

Figure 3: A Spiral Source

Note that the vector fields shown in these figures determine only the direction of the next iteration at a given point, and not the size of the iterative derivative step (which in many cases lands off screen).

Example 1.3 *If we modify example 1.1 and change one sign to become:*

$$\begin{cases} x_{n+1} = -1.05x_n \\ y_{n+1} = 1.1y_n \end{cases}$$

or

$$\vec{Y}_{n+1} = \begin{pmatrix} -1.05 & 0 \\ 0 & 1.1 \end{pmatrix} \vec{Y}_n$$

*then for this system, the eigenvalues are -1.05 and 1.1; hence the origin is still a source. The corresponding eigenlines remain the $x-$ axis and the $y-$ axis, but because one eigenvalue is negative, the iteration flips between the first and second quadrant of the plane (or the third and fourth, depending on the initial condition), that is, back and forth across an eigenline. The origin in this case is called a **flip source**.(See Figure 4).*

Example 1.4 *Now if we consider the system*

$$\begin{cases} x_{n+1} = -1.05x_n \\ y_{n+1} = -1.1y_n \end{cases}$$

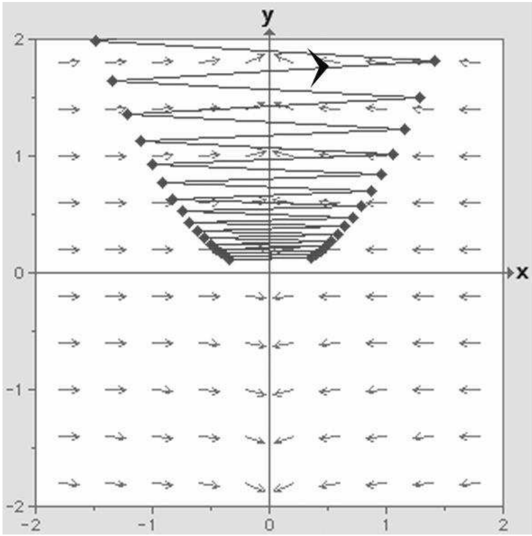

Figure 4: A Flip Source

or

$$\vec{Y}_{n+1} = \begin{pmatrix} -1.05 & 0 \\ 0 & -1.1 \end{pmatrix} \vec{Y}_n$$

then we find that both eigenvalues are negative, and the iteration flips either between the first and third quadrant, or the second and fourth, that is across both eigenlines. The origin in this case is a ***double flip source****.(See Figure 5).*

2 THE TRACE-DETERMINANT PLANE

In the preceding section we have seen that, as a source, the equilibrium solution $(0, 0)$ comes in 4 different types. One wishes therefore to find ways to classify this solution and consequently predict the behavior of initial values problems. It is necessary therefore to pause and review the big picture! For 2×2 linear systems of differential equations, parabola $det(A) = \frac{1}{4}(tr(A))^2$ is the determining factor for the existence of two distinct eigenvalues, one repeated eigenvalue, or complex eigenvalues for the system matrix A. On this issue, Paul Blanchard, Robert Devaney, and Glenn Hall write: ”As is often the case in mathematics, it is often helpful to view information in several different ways. Since we are looking for the ”big picture,” why not try to summarize the different behaviors for linear systems in a picture rather than a table? One such picture is the *trace-determinant plane*” ([1], p. 333). The same is also true for 2×2 linear iterative systems. Indeed, the eigenvalues of such systems are solutions to the characteristic equation of A given by: $\lambda^2 - tr(A)\lambda + det(A) = 0$. Thus, in case $(tr(A))^2 - 4det(A) < 0$ (i.e. above the parabola $det(A) = \frac{1}{4}(tr(A))^2$ in the trace-determinant plane), the iterated points will either spiral toward or away from the origin (or simply form closed loops if $|\lambda| = 1$). On the other hand, below the parabola ($(tr(A))^2 - 4det(A) > 0$), the origin is in general either a source, a sink, or a saddle, a regular source, or a regular sink.

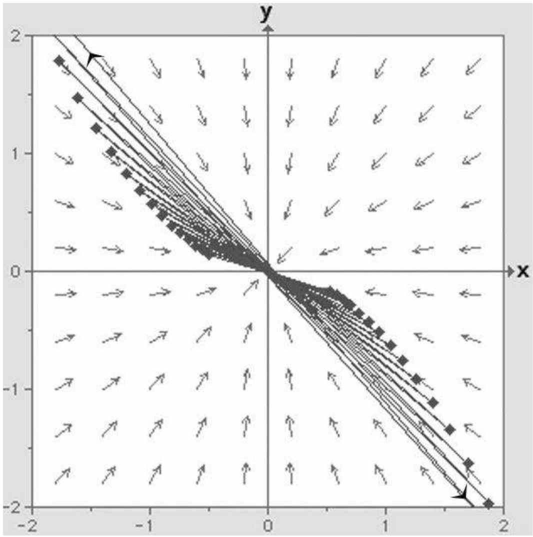

Figure 5: A Double Flip Source

The theorem below identifies more specifically these regions. We only present the proof of the first part.

Theorem 2.1 *Let $\vec{Y}_{n+1} = A\vec{Y}_n$, where $\vec{Y}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and A is a 2×2 matrix having two distinct eigenvalues λ_1 and λ_2. The two diagonal lines*

$$det(A) = tr(A) - 1 \text{ and } det(A) = -tr(A) - 1$$

identify the regions where we have a sink, a source, or a saddle as follows:

1. *If $|det(A) + 1| < |tr(A)|$, then the origin is a saddle point.*
2. *If $det(A) > tr(A) - 1$ and $det(A) > -tr(A) - 1$, then the origin is a source whenever $det(A) > 1$, and a sink whenever $-1 < det(A) < 1$.*

Proof: 1. Suppose that $tr(A) > 0$, then $|det(A) + 1| < tr(A)$ or $-tr(A) < det(A) + 1 < tr(A)$. Since $det(A) + 1 < tr(A)$, then $\lambda_1\lambda_2 - \lambda_1 - \lambda_2 + 1 < 0$ or $(\lambda_1 - 1)(\lambda_2 - 1) < 0$. This fact implies that either

$$\lambda_1 < 1 \text{ and } \lambda_2 > 1, \text{ or } \lambda_1 > 1 \text{ and } \lambda_2 < 1.$$

Since $det(A) + 1 > -tr(A)$, then by a similar argument, either

$$\lambda_1 > -1 \text{ and } \lambda_2 > -1 \text{ or } \lambda_1 < -1 \text{ and } \lambda_2 < -1.$$

In conclusion, we either have:

$$-1 < \lambda_1 < 1, \lambda_2 > 1$$

or

$$-1 < \lambda_2 < 1, \lambda_1 > 1.$$

Thus, the origin is a saddle.
A similar argument shows that if $tr(A) < 0$, then either

$$-1 < \lambda_1 < 1, \lambda_2 < -1$$

or

$$-1 < \lambda_2 < 1, \lambda_1 < -1$$

resulting again in a saddle. □

The line segment $det(A) = 1$ inside the parabola separates the spiral sinks from the spiral sources. Clearly, in order for this *bifurcation* to occur, the points on this line are expected to be *centers*. Indeed, when $det(A) = 1$ the complex eigenvalues take the form $\lambda = \dfrac{tr(A) \pm i\sqrt{4 - tr(A)^2}}{2}$ and hence $||\lambda||^2 = \frac{1}{4}\left(tr(A)^2 + (4 - tr(A)^2)\right) = 1$.

The qualitative behavior for solutions of iterative systems that correspond to points in the source/sink/saddle/center regions of the Trace-Determinant plane has already been studied (see [2] and [4]). Figure 6 summarizes this behavior. The authors however are not aware of any literature on the subject describing the qualitative behavior on the boundaries of these regions, specifically the parabola and the diagonals. In this paper, we fill in this gap and describe also in detail the behavior on the trace axis, the determinant axis, and the center axis.

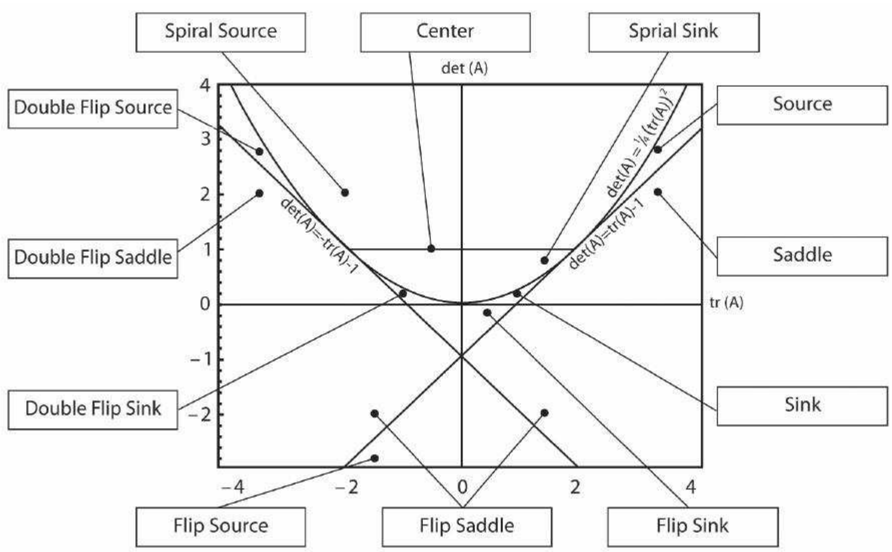

Figure 6: The Source/Sink/Saddle/Center Regions in the Trace Determinant Plane

3 The Borderline Cases

We now look at the system $\vec{Y}_{n+1} = A\vec{Y}_n$ on the borders of the regions as seen in Figure 6. One important border is the **parabola** given by $det(A) = \frac{1}{4}(tr(A))^2$. In this case, the characteristic equation has a double root. The general form of the solution has been investigated by the authors in [3] and the following was proved:

Theorem 3.1 *Consider the system $\vec{Y}_{n+1} = A\vec{Y}_n$ having one repeated eigenvalue $\lambda \neq 0$, and let $\vec{v}$ denote a corresponding eigenvector.*

1. *If a second independent eigenvector exists for the repeated eigenvalue λ, then the general solution of the system takes the form $Y_n = \lambda^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, where x_0, y_0 are some initial conditions.*

2. *If another independent eigenvector does not exist, then the general solution of the system takes the form:*
$$\vec{Y}_n = c_1\lambda^n\vec{v} + c_2\lambda^n\left(n\vec{v} + \vec{u}\right)$$
where $\left(A - \lambda I\right)\vec{u} = \lambda\vec{v}$.

Figures 7 and 8 illustrate graphically the phase portrait in each case.

We now wish to investigate the phase portraits on the **right diagonal** $det(A) = tr(A) - 1$. In this case, we can easily show from the characteristic equation $\lambda^2 - tr(A)\lambda + det(A) = 0$ that one eigenvalue is always 1. Let L_1 denote its corresponding eigenline. The second eigenvalue can vary; indeed, $\lambda = \dfrac{tr(A) \pm |tr(A) - 2|}{2}$. The general solution takes the form $\vec{Y}_n = c_1\vec{v_1} + c_2(\lambda_2)^n\vec{v_2}$. Since one part of the solution is fixed ($c_1\vec{v_1}$), $\vec{Y}_n$ will always form a line parallel to the eigenline L_2 corresponding to the eigenvector $\vec{v_2}$. Three subcases arise:

1. If $tr(A) > 2$, then $\lambda_1 = 1$ and $\lambda_2 = tr(A) - 1$. In fact in this case, $\lambda_2 > 1$ and the general solution diverges along a line parallel to L_2.

2. If $tr(A) = 2$, then $\lambda_2 = 1$, but also $det(A) = \frac{1}{4}\left(tr(A)\right)^2$ so this point is also on the parabola, a repeated eigenvalue vase investigated in [2].

3. If, on the other hand, $tr(A) < 2$, then $\lambda_1 = 1$ and $\lambda_2 = tr(A) - 1 < 1$. Thus the general solution depends on one of five cases:

   (a) If $0 < \lambda_2 < 1$, then the general solution $\vec{Y}_n$ converges to the point $c_1\vec{v_1}$ on the eigenline L_1.

   (b) If $\lambda_2 = 0$, then $tr(A) = 1, det(A) = 0$ (a point that also belongs to the trace axis), and the general solution is simply $\vec{Y}_n = c_1\vec{v_1}$; that is, starting at any initial condition, the iteration maps it to a point on eigenline L_1.

   (c) If $-1 < \lambda_2 < 0$, then the general solution $\vec{Y}_n$ converges to the point $c_1\vec{v_1}$ on the eigenline L_1 but flips while doing so because of the negative eigenvalue.

   (d) If $\lambda_2 = -1$, then $tr(A) = 0, det(A) = -1$ (a point that belongs also to the determinant axis), and $\vec{Y}_n = c_1\vec{v_1} + c_2(-1)^n\vec{v_2}$. Hence the iteration flips between $c_1\vec{v_1} + c_2\vec{v_2}$ and $c_1\vec{v_1} - c_2\vec{v_2}$.

   (e) Finally if $\lambda_2 < -1$, then $\vec{Y}_n$ flips and diverges along a line parallel to L_2.

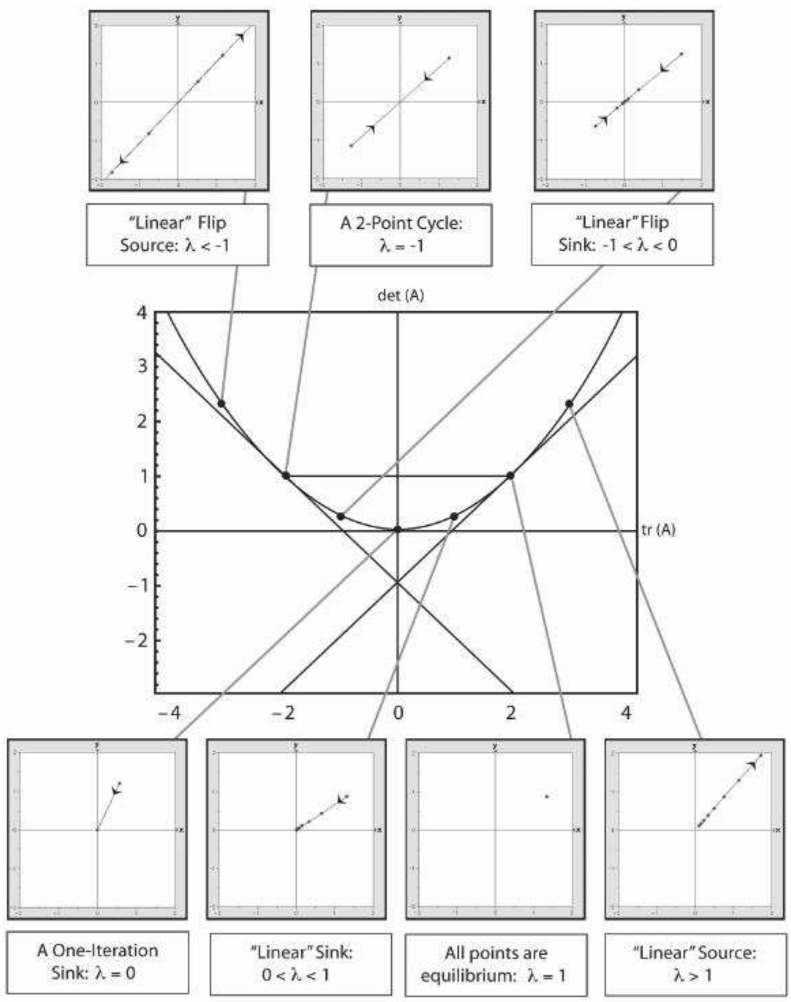

Figure 7: The Case of Two Independent Eigenvectors

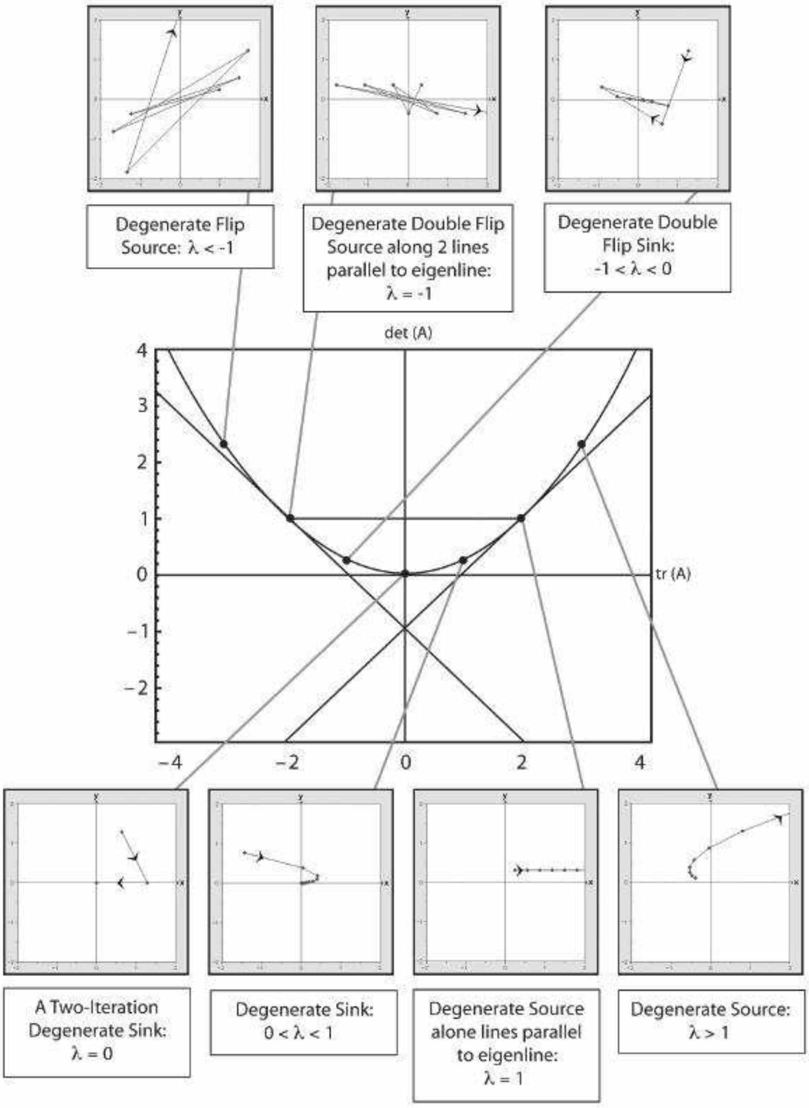

Figure 8: The Case of One Eigenvector

Figure 9 summarizes all the possibilities. In this figure however, the system matrix has been diagonalised so that eigenline L_1 corresponding to the eigenvalue 1 is the x-axis.

On the **left diagonal** $det(A) = -tr(A) - 1$, one can easily show that an eigenvalue is -1 (with corresponding eigenline L_1), while the second one varies. Indeed, $\lambda = \dfrac{tr(A) \pm |tr(A) + 2|}{2}$, and the general solution is $\vec{Y_n} = c_1(-1)^n\vec{v_1} + c_2(\lambda_2)^n\vec{v_2}$. The following sub-cases arise:

1. If $tr(A) > -2$, then $\lambda_1 = -1, \lambda_2 = tr(A) + 1 > -1$. Thus the general solution depends on one of five cases:
   (a) If $-1 < \lambda_2 < 0$, then both eigenvalues are negative. The iteration double flips to converge to a cycle of two points $c_1\vec{v_1}$ and $-c_1\vec{v_1}$.
   (b) If $\lambda_2 = 0$, then $det(A) = 0$ (a point on the trace axis), and the iteration flips between $c_1\vec{v_1}$ and $-c_1\vec{v_1}$ along L_1.
   (c) If $0 < \lambda_2 < 1$, then $\vec{Y_n}$ flips initially but eventually converges to a 2-point cycle $c_1\vec{v_1}$ and $-c_1\vec{v_1}$ along L_1.
   (d) If $\lambda_2 = 1$, then $\vec{Y_n} = c_1(-1)^n\vec{v_1} + c_2\vec{v_2}$, which is a solution that flips between $c_1\vec{v_1} + c_2\vec{v_2}$ and $-c_1\vec{v_1} + c_2\vec{v_2}$.
   (e) Finally if $\lambda_2 > 1$, then $\vec{Y_n}$ diverges and flips along two lines parallel to the second eigenline L_2.
2. If $tr(A) < -2$, then $\lambda_2 < -1$, and the iteration double flips and diverges.
3. If $tr(A) = -2$, then $\lambda_1 = \lambda_2$, a repeated eigenvalue case investigated in [3].

Figure 10 summarizes all the possibilities. In this figure however, the system matrix has been diagonalised so that the eigenline L_1 corresponding to the eigenvalue -1 is the x-axis.

On the **determinant axis** the trace is equal to zero; hence the eigenvalues take the form $\lambda = \frac{\pm\sqrt{-4det(A)}}{2}$. Solutions are therefore either real or pure imaginary. Moving from the upper-half of the trace-determinant plane until we reach the trace axis, solutions will begin by spiralling out, then bifurcate at the center axis to spiral inwards to the origin. In the lower half of the plane, the origin moves from a flip sink to a flip source, with a bifurcation occurring when $det(A) = -1$. In this latter case, the eigenvalues are $\lambda = \pm 1$, hence the general solution takes the form $\vec{Y_n} = c_1\vec{v_1} + c_2(-1)^n\vec{v_2}$, and the iterated points flip between two fixed points $c_1\vec{v_1}+c_2\vec{v_2}$ and $c_1\vec{v_1}-c_2\vec{v_2}$. Figure 11 summarizes all the possibilities. Again the system matrix has been diagonalised so that the eigenlines are the x- and y-axes.

On the **trace axis** the determinant of the matrix is zero; hence one eigenvalue λ_1 is necessarily zero, and the general solution of the system takes the form $\vec{Y_n} = c\lambda_2^n\vec{v}$. Consequently, the first time the iteration is applied, the initial point (x_0, y_0) is mapped to a point on the eigenline L_2 corresponding to the non-zero eigenvector λ_2. Thus all subsequent iterations remain on that line. More precisely,

1. If $\lambda_2 > 1$, the iteration diverges along eigenline L_2.

2. If $\lambda_2 = 1$, then $Y_n = c\vec{v}$ and the iteration is fixed at that point. Notice that the corresponding point in the Trace-Determinant plane $(tr(A) = 1, det(A) = 0)$ is also a point on the right diagonal.

3. If $0 < \lambda_2 < 1$, then the iteration converges to the origin along L_2.

4. If $-1 < \lambda_2 < 0$, then the iteration converges to the origin but this time it flips along L_2.

5. If $\lambda_2 = -1$, then the solution takes the form $\vec{Y}_n = c(-1)^n\vec{v}$, and the iteration flips along L_2 between $c\vec{v}$ and $-c\vec{v}$. This case is special since the point $(tr(A) = -1, det(A) = 0)$ is also a point on the left diagonal.

6. Finally, if $\lambda_2 < -1$, then the iteration flips and diverges along L_2.

Figure 12 summarizes all the possibilities. The system matrix has been diagonalised so that L_2 is the x-axis.

The **center axis** exhibits some exceptional behavior at various points. As we move along the center axis starting at $(2, 1)$ and ending at $(-2, 1)$, we look in particular at the points $(1, 1)$, $(0, 1)$ and $(-1, 1)$.

Proposition 3.2 *1. If $det(A) = 1$ and $tr(A) = 1$, then the iteration forms a closed loop of 6 points.*

2. If $det(A) = 1$ and $tr(A) = 0$, then the iteration forms a closed loop of 4 points.

3. If $det(A) = 1$ and $tr(A) = -1$, then the iteration froms a closed loop of 3 points.

Proof:

1. When $det(A) = 1$ and $tr(A) = 1$, $\lambda = \dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$, and the general complex solution takes the form $\vec{Y}_n = c_1\lambda_1^n\vec{v_1} + c_2\overline{\lambda_1}^n\vec{v_2}$. Since $\lambda_1^3 = -1$ and $\lambda_1^6 = 1$, the iterated points form a 6-point cycle.

2. When $tr(A) = 0$, the eigenvalues are $\lambda = \pm i$. Since $i^4 = 1$, the iteration forms a 4-point cycle.

3. Finally, when $tr(A) = -1$, $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. In this case $\lambda^3 = 1$, and the iteration thus forms a 3-point cycle.

□

Figure 13 shows various possibilities.
The cases exhibited in Proposition 3.2 are not the only possible cycles. One can show [1] that if the system matrix is a rotation matrix of the form $A = \begin{pmatrix} Cos\left(\frac{2\pi}{k}\right) & -Sin\left(\frac{2\pi}{k}\right) \\ Sin\left(\frac{2\pi}{k}\right) & Cos\left(\frac{2\pi}{k}\right) \end{pmatrix}$, where k is a positive integer, then the iterated points form a k-point cycle. Thus if for instance $k = 6$, the iteration is

[1]We wish to thank Beverly H. West of Cornell University, and Bj∅rn Ferslager of Haslev Gymnasium, Denmark, for their valuable suggestion.

expected to form a 6-point cycle. Indeed, in this case, the system matrix is $A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, consequently, $tr(A) = 1$, and $det(A) = 1$. In the previous proposition we showed that a 6-point cycle is formed in this case.

In conclusion, this paper not only provides a complete linear classification of 2×2 linear iterative systems, but also a visual illustration of this classification is presented using a software that was particularly developed for that purpose. Technology has been the driving force for many new investigations in mathematics, and indeed the work developed by Hohn in projects like IDE encouraged the authors to pursue the idea of a similar software for iterative systems. The software is not as sophisticated as IDE, but its dynamical environment lead the authors to conjecture many results before proving them analytically.

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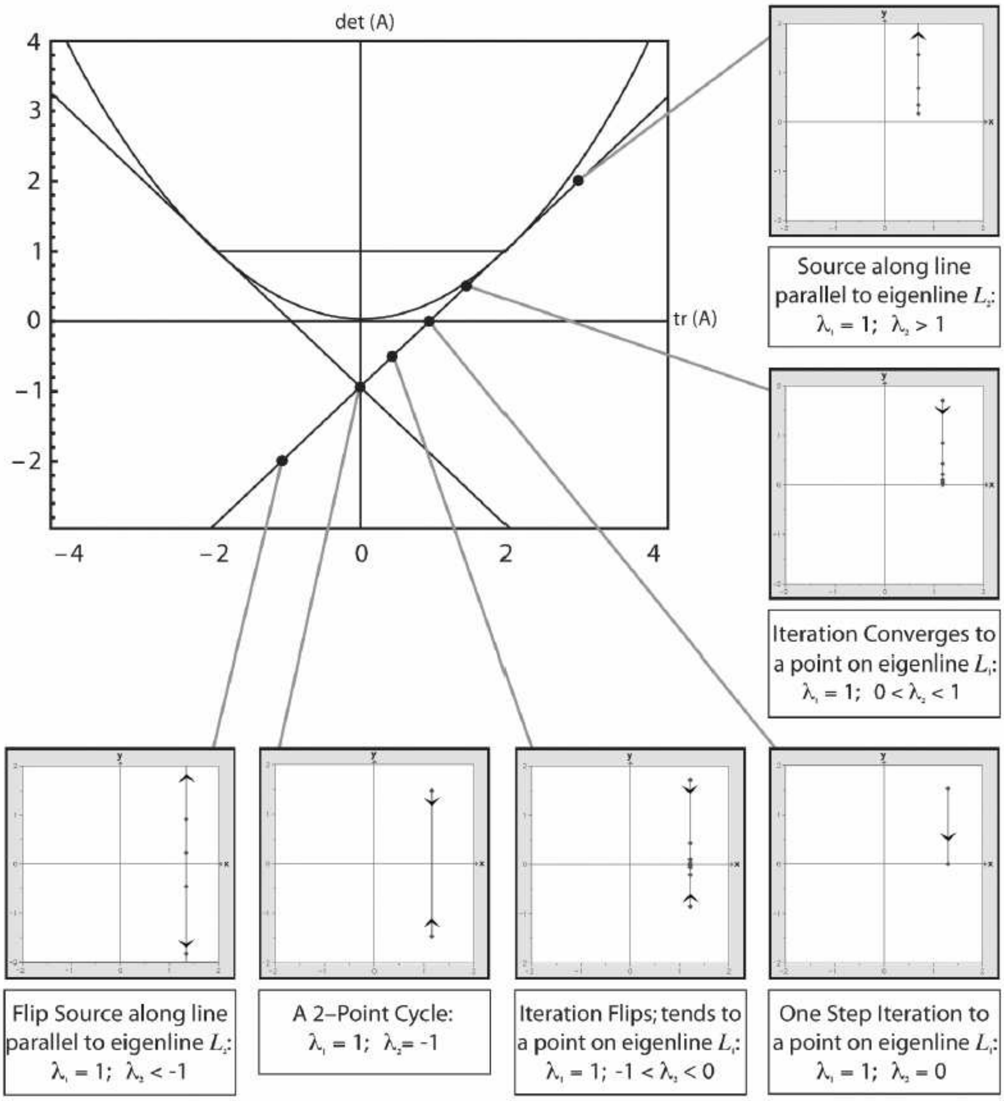

Figure 9: The Right Diagonal

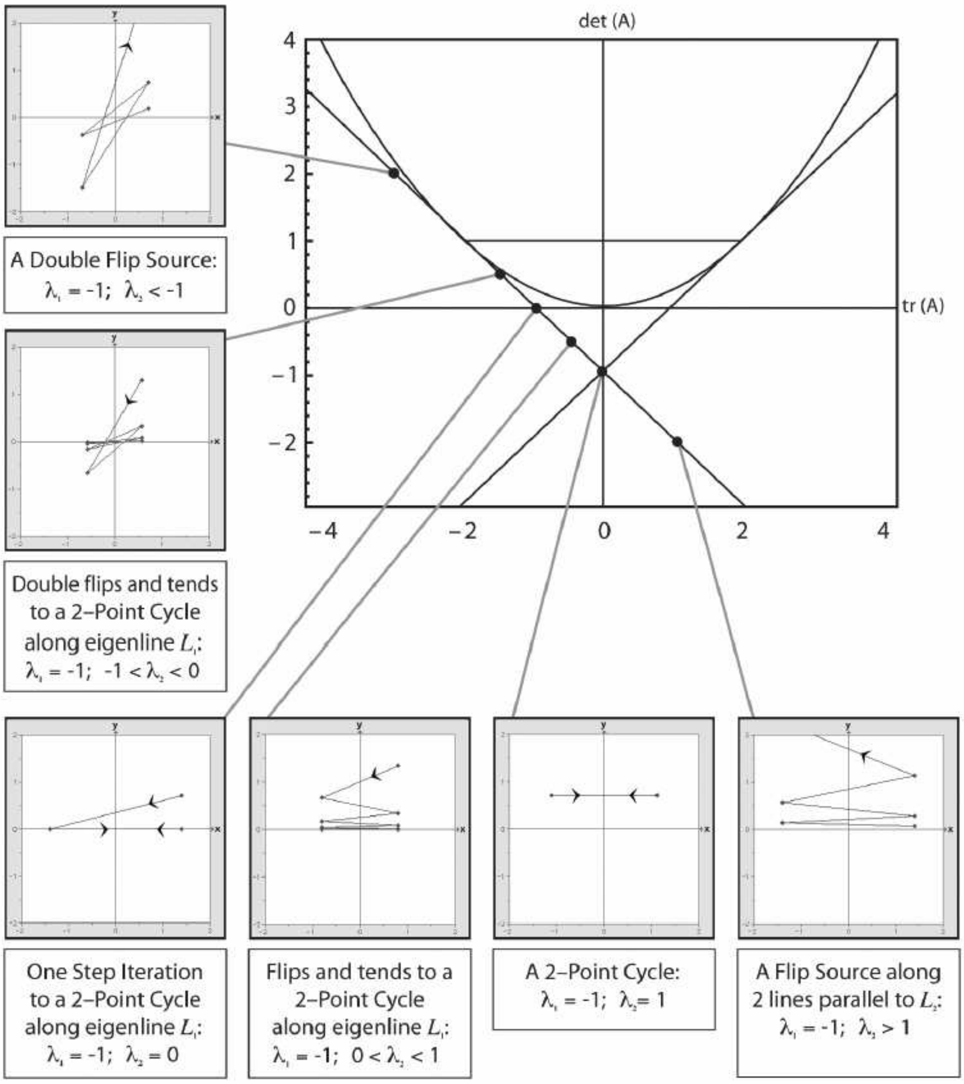

Figure 10: The Left Diagonal

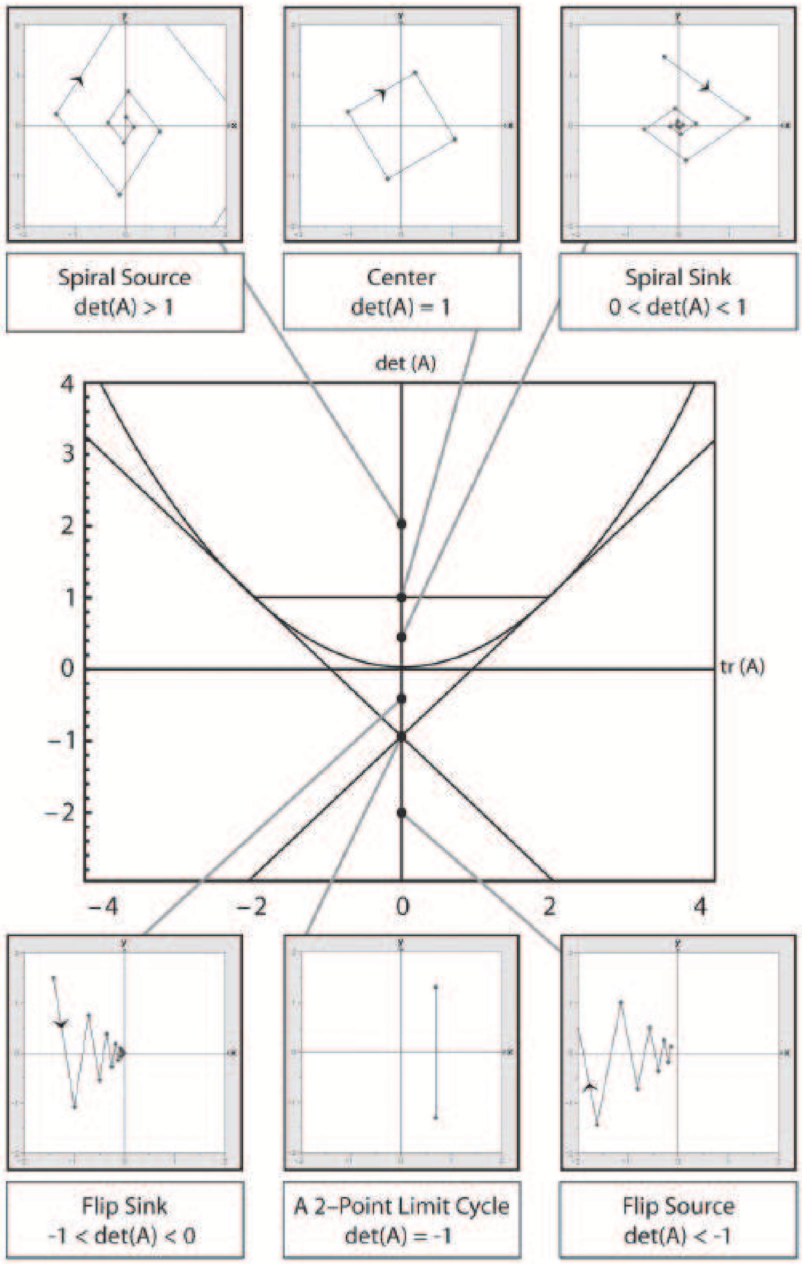

Figure 11: The Determinant Axis

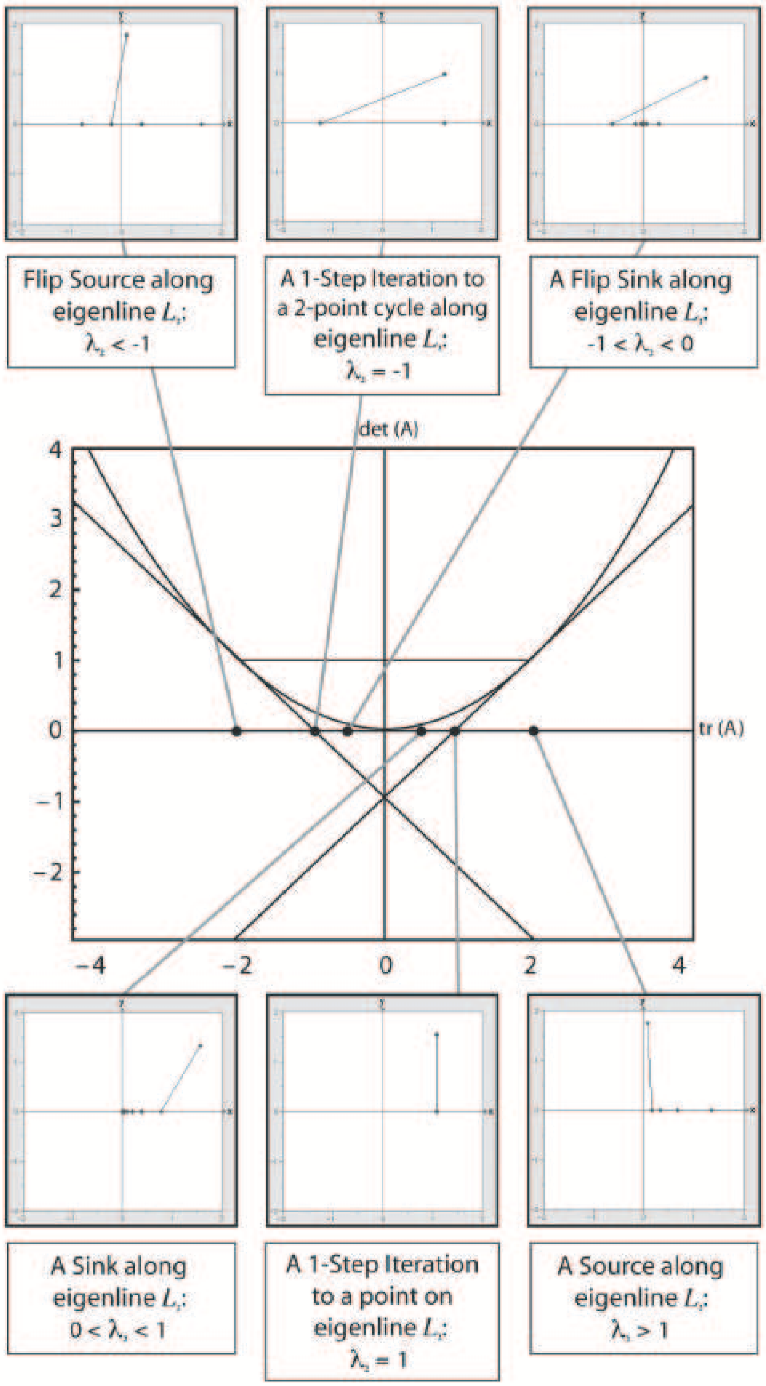

Figure 12: The Trace Axis

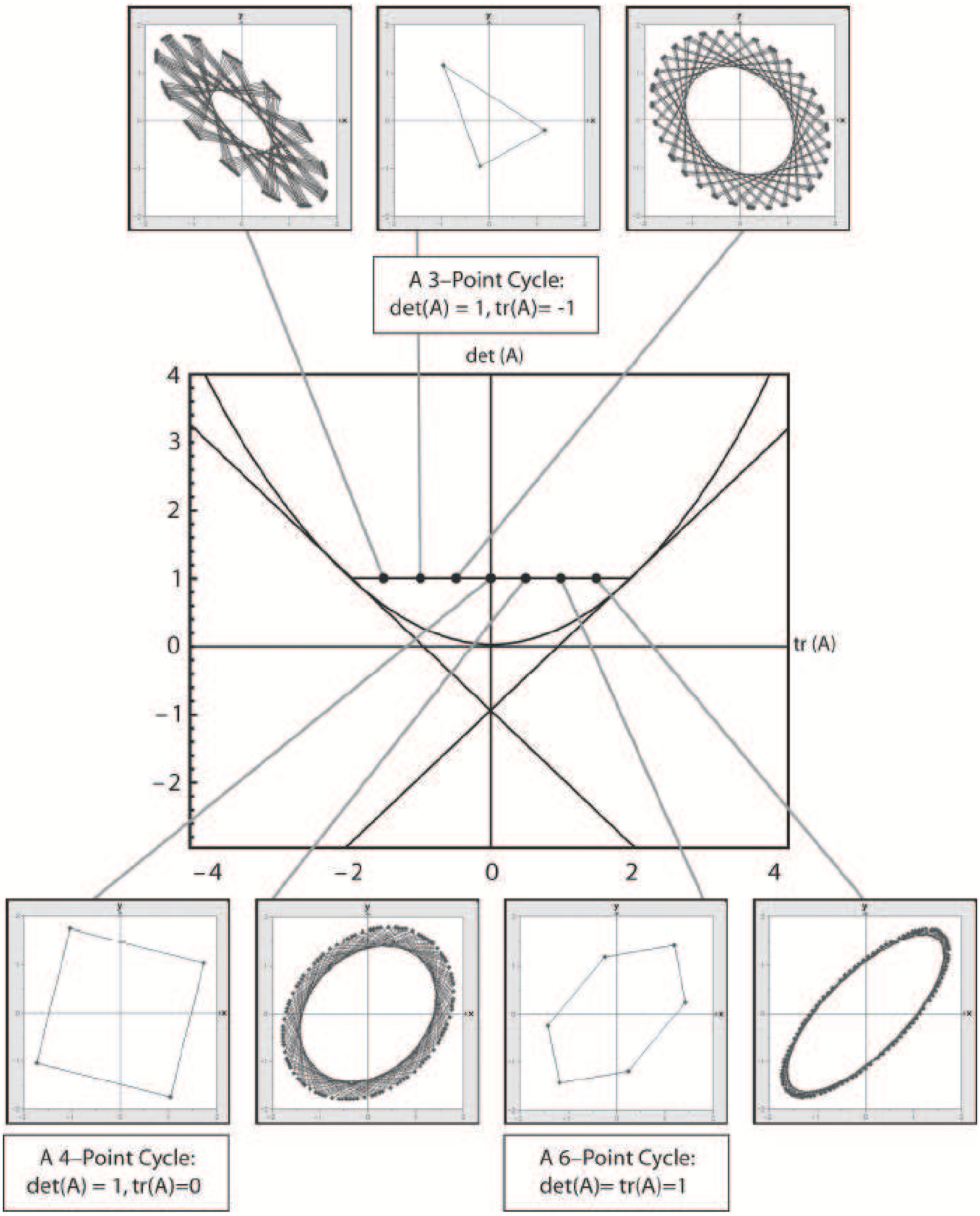

Figure 13: The Center Axis